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On the Limit Distribution of Multiscale Test Statistics for Nonparametric Curve Estimation

Lutz Dümbgen, Vladimir I. Piterbarg* and Dmitrii Zholud†
Bern University and Lomonosov Moscow State University

Abstract
We prove continuity of the limit distribution function of certain multiscale test statistics which are used in nonparametric curve estimation.

A particular variant of multiscale testing was introduced in Dümbgen and Spokoiny (2001) in order to test qualitative hypotheses about an unknown regression function such as nonpositivity, monotonicity or concavity. These authors considered a continuous white noise model, and their tests involved test statistics of the form

$$T^0 \;=\; \sup_{s,t\in[0,1]:0<t-s\le a} \left(\frac{1}{\sqrt{t-s}} \left| \int_0^1 \beta\Big(\frac{x-s}{t-s}\Big)\, dW(x) \right| - \sqrt{2\ln\frac{K}{t-s}} \right),$$

where W is the standard Wiener process, $0 < a \le 1$, $K \ge 1$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a certain test signal with finite total variation such that $\int_0^1 \beta(x)^2 dx = 1$ and $\beta(t) = 0$ for $t \notin [0,1]$. Later on, in Dümbgen (2002), Dümbgen and Johns (2004) and Dümbgen (2001), these methods have been extended to more traditional regression models, and the test statistic T^0 above appeared only as the distributional limit of multiscale rank or sign statistics. In order to deduce convergence of arbitrary quantiles from weak convergence, the continuity of the distribution function of T^0 is crucial. Showing that T^0 has support $[0,\infty)$ is not very difficult, and various Monte Carlo simulations indicated that the distribution function of T^0 is indeed continuous. The present paper verifies the latter conjecture for a large class of test signals β. Precisely, we assume a special behavior of β near the end points of $[0,1]$ by requiring that

$$\int_{[0,1]\cap[t,t+1]} \beta(x)\beta(x-t)\, dx \;=\; 1 - C|t|^\alpha + o(|t|^\alpha) \tag{1}$$

as $t \to 0$, where $C > 0$, $0 < \alpha < 2$. Note that this condition is satisfied in case of $\beta(x) = 1_{[0,1]}(x)(2x-1)$ and $\beta(x) = 1_{[0,1]}(x)x^\gamma$ with $\gamma = 0, 1$. These test signals appear in Dümbgen (2002) and Dümbgen and Johns (2004). Condition (1) is violated if β is too smooth. Now we formulate the desired result.

Theorem 1. *Let condition (1) be fulfilled. Then the distribution function of T^0 is continuous.*

To prove this theorem, we begin with the maximum of a Gaussian process,

$$\xi \;=\; \sup_{s,t\in[0,1]:0<t-s\le a} \left(\frac{1}{\sqrt{t-s}} \int_0^1 \beta\Big(\frac{x-s}{t-s}\Big)\, dW(x) - \sqrt{2\ln\frac{K}{t-s}} \right).$$

We prove that $P(\xi > 0) = 1$ and 0 is the starting point of the distribution of ξ. From here it will follow that the distribution of ξ has no atoms, and then so for T^0.

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†E-mail: dmitrii@chalmers.se

Lemma 1. *The probability that*

$$\sup_{s\in[0,1-u]} u^{-1/2}\int_0^1 \beta\left(\frac{x-s}{u}\right)dW(x) \;>\; \sqrt{2\ln K/u}$$

infinitely often (i.o.) as $u\downarrow 0$ equals one.

To prove this lemma we will need the following theorem of J. Pickands, see Leadbetter et al. (1983).

Theorem 2. *Let $X(t)$ be an a.s. continuous stationary zero mean Gaussian process, with covariation function $r(t)$, satisfying the following conditions,*

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \text{ as } t\to 0, \text{ and } r(t)\ln t \to 0 \text{ as } t\to\infty,$$

where $C>0$, $0<\alpha\le 2$. Then, for any x,

$$P\left(a_t(M(t)-b_t)\le x\right) \to e^{-e^{-x}} \text{ as } t\to\infty,$$

where

$$M(t) = \sup_{0\le s\le t} X(s), \quad a_t = \sqrt{2\ln t}, \text{ and}$$
$$b_t = \sqrt{2\ln t} + \frac{1}{\sqrt{2\ln t}}\left(\frac{2-\alpha}{2\alpha}\ln\ln t + \ln(C^{1/\alpha}(2\pi)^{-1/2}H_\alpha 2^{(2-\alpha)/2a})\right),$$

H_α is a positive constant (Pickands' constant; in particular, $H_1=1$) .

Proof of Lemma 1: Let

$$A = P\left(\sup_{s\in[0,1-u]} u^{-1/2}\int_0^1 \beta\left(\frac{x-s}{u}\right)dW(x) > \sqrt{2\ln K/u} \text{ i.o. as } u\to 0\right).$$

Introducing new variables $z=(x-s)/u$ and $t=s/u$ we get

$$A = P\left(\sup_{t\in[0,1/u-1]} u^{-1/2}\int_0^1 \beta(z)dW(u(z+t)) > \sqrt{2\ln K/u} \text{ i.o. as } u\to 0\right).$$

Now, denoting $T=1/u-1$, we find that

$$A = P\left(\sup_{t\in[0,T]} X_T(t) > \sqrt{2\ln K(T+1)} \text{ i.o. as } T\to\infty\right),$$

where

$$X_T(t) = (T+1)^{1/2}\int_0^1 \beta(z)dW((z+t)/(T+1)),\ t\in[0,T].$$

It is immediate that $X_T(t)$ is a stationary zero mean Gaussian process with covariation function $r(t)=\int_{[0,1]\cap[t,t+1]}\beta(x)\beta(x-t)dx$. Introduce intervals $U_n=[e^{2p_n}, 2e^{2p_n}-1]$ and $T_n = 2^n e^{2p_n}-1$, where the sequence $\{p_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty}p_n=\infty$ will be defined later.

Examine now the events $A_n = \{\sup_{t\in U_n} X_{T_n}(t) > \sqrt{2\ln(K(T_n+1))}\}$. First, the events $\{A_n\}_{n=1}^\infty$ are mutually independent because the processes $\{X_{T_i}(t),\ t\in U_i\}_{i=1}^\infty$ are obtained from increments of the Wiener process on the non-intersecting sub-intervals $[2^{-i}, 2^{-i+1})$.

Further, since X_T is stationary,

$$\begin{aligned} P(A_n) &= P\left(\sup_{t\in U_n} X_{T_n}(t) > \sqrt{2\ln K(T_n+1)}\right) \\ &= P\left(\sup_{t\in[0,e^{2^{p_n}}-1]} X(t) > \sqrt{2\ln K(T_n+1)}\right) \\ &= P\left(\sup_{t\in[0,t_n]} X(t) > \sqrt{2\ln K(T_n+1)}\right), \end{aligned}$$

where $t_n = e^{2^{p_n}} - 1$, and $X(t) = \int_0^1 \beta(y)dW(y+t)$ is a stationary process with the same distribution as X_T. Obviously X satisfies the conditions of Theorem 2 with C and α as in assumption (1). Furthermore,

$$\begin{aligned} b_{t_n} &= \sqrt{2\ln(t_n)} + \frac{1}{\sqrt{2\ln(t_n)}}\left(\frac{2-\alpha}{2\alpha}\ln\ln t_n + \ln\left(\frac{C^{1/\alpha}H_\alpha 2^{(2-\alpha)/2\alpha}}{\sqrt{2\pi}}\right)\right) \\ &= \sqrt{2\ln t_n} + \frac{1}{\sqrt{2\ln t_n}}\left(\frac{2-\alpha}{2\alpha}\ln\ln t_n + C_1\right), \end{aligned}$$

where $C_1 = C_1(\alpha, C) = \ln(C^{1/\alpha}H_\alpha 2^{(2-\alpha)/2\alpha}/\sqrt{2\pi})$. Therefore,

$$\begin{aligned} & a_{t_n}\big(\sqrt{2\ln K(T_n+1)} - b_{t_n}\big) \\ &= \sqrt{2\ln t_n}\left(\sqrt{2\ln K(T_n+1)} - \sqrt{2\ln t_n}\right) - \left(\frac{2-\alpha}{2\alpha}\ln\ln t_n + C_1\right) \\ &= \sqrt{2\ln t_n}\left(\frac{2\ln K(T_n+1) - 2\ln t_n}{\sqrt{2\ln K(T_n+1)} + \sqrt{2\ln t_n}}\right) - \left(\frac{2-\alpha}{2\alpha}\ln\ln t_n + C_1\right), \end{aligned}$$

and since $\ln K(T_n+1) = (1+o(1))\ln t_n$, the expression above equals

$$\begin{aligned} & \ln\frac{K(T_n+1)}{t_n}(1+o(1)) - \left(\ln\ln t_n^{(2-\alpha)/2\alpha} + C_1\right) \\ &= \ln\left(\frac{K(T_n+1)}{t_n\ln t_n^{(2-\alpha)/2\alpha}}(1+o(1))\right) - C_1 \\ &= \ln\left(\frac{K2^n e^{2^{p_n}}(1+o(1))}{(e^{2^{p_n}}-1)\ln(e^{2^{p_n}}-1)^{(2-\alpha)/2\alpha}}\right) - C_1 \\ &= \ln\left(\frac{K2^n(1+o(1))}{(2^{p_n}+o(1))^{(2-\alpha)/2\alpha}}\right) - C_1 \\ &= \ln(K2^{n-p_n(2-\alpha)/2\alpha}(1+o(1))) - C_1. \end{aligned}$$

Now, choosing $p_n = 2n\alpha/(2-\alpha)$, the latter expression

$$\begin{aligned} & (n - p_n(2-\alpha)/2\alpha)\ln 2 + \ln K - C_1 + o(1) \\ &= \ln K - C_1 + o(1) \ \to\ L < \infty, \end{aligned}$$

and thus

$$\begin{aligned} P(A_n) &= P\left(a_{t_n}(M(t_n) - b_{t_n}) > a_{t_n}(\sqrt{2\ln K(T_n+1)} - b_{t_n})\right) \\ &\geq P\left(a_{t_n}(M(t_n) - b_{t_n}) > L\right). \end{aligned}$$

According to Theorem 2,

$$P(A_n) = P\left(a_{t_n}(M(t_n) - b_{t_n}) > L\right) \to 1 - e^{-e^{-L}} \text{ as } n \to \infty,$$

and for any $\epsilon > 0$, one can find N such that $P(A_n) > 1-e^{-e^{-K}}-\epsilon > 0$ for all $n > N$. Thus we get that $\sum_{n=1}^{\infty} P(A_n) = \infty$, and applying the Borell-Cantelly Lemma yields that $P(A_n \text{ i.o. as } n \to \infty) = 1$. Finally, $U_n \subset [0, T_n]$ and since $T_n \to \infty$ as $n \to \infty$ we get that $P(A) = 1$. $\square$

Lemma 2. *The distribution of ξ has no positive atoms.*

Proof: Let the random process $(\Psi(u))_{u\in[0,1]}$ be defined by

$$\Psi(u) \;=\; \sup_{s\in[0,1-u]} u^{-1/2} \int_0^1 \beta\left(\frac{x-s}{u}\right) dW(x) - \sqrt{2\ln K/u}$$

for $u > 0$, and $\Psi(0) = 0$. It follows from Theorem 2.1 of Dümbgen and Spokoiny (2001) that Ψ is continuous at zero. The latter entails that for any fixed $\varepsilon > 0$,

$$P\left(\sup_{u\in[0,u_0]} \Psi(u) \geq \varepsilon\right) \to 0 \tag{2}$$

as $u_0 \to 0$. Now consider the Gaussian process

$$Y(u, s) = u^{-1/2} \int_0^1 \beta\left(\frac{x-s}{u}\right) dW(x) - \sqrt{2\ln K/u},\; u \in [u_0, a],\; s \in [0, 1-u].$$

This process has positive variance, bounded expectation and continuous sample paths. Therefore by Proposition 11.4, Davydov et al. (1998), the distribution of its supremum has no atoms. Further,

$$\xi = \max\left\{\max_{u,s} Y(u, s), \zeta\right\},$$

where

$$\zeta = \sup_{u\in[0,u_0]} \Psi(u)\,.$$

For any fixed $x > 0$,

$$P(\xi = x) \;\leq\; P\left(\max_{u,s} Y(u, s) = x\right) + P(\zeta = x) \;=\; P(\zeta = x) \;\to\; 0$$

as $u_0 \to 0$, whence $P(\xi = x) = 0$.

Proof of Theorem 1: With Ψ as in the proof of Lemma 2, it follows from $\Psi(u) \to 0$ as $u \downarrow 0$ that $\xi = \sup_{u\in(0,a]} \Psi(u) \geq 0$. According to Lemma 2, ξ has no atom on $(0,\infty)$, and by Lemma 1,

$$\begin{aligned} 1 &= P\left(\Psi(u) > 0 \text{ i.o. as } u \downarrow 0\right) \\ &\leq P\left(\exists\, u \in (0, 1] \;:\; \Psi(u) > 0\right) \\ &= P(\xi > 0), \end{aligned}$$

so that the distribution of ξ has no atom at zero, and, therefore it is continuous. By symmetry, $T^0 = \max(\xi, \xi')$, where ξ' has the same distribution as ξ. Thus $P(T^0 = x) \leq P(\xi = x)+P(\xi' = x) = 0$, and this completes the proof of our Theorem.

Remark 1: Despite the fact that Theorem 2 permits $\alpha = 2$, the strong inequality essential for the proof of divergency of $\sum_{n=1}^{\infty} P(A_n)$ does not hold when $\alpha = 2$.

Remark 2: For the simpler functional

$$T_1 = \sup_{1 \geq t > s \geq 0} \left(\left| \frac{W(t) - W(s)}{\sqrt{t-s}} \right| - \sqrt{2 \ln \frac{1}{t-s}} \right)$$

the assertion of Theorem 1 follows from Theorem B, Reves (1982) and Lemma 2.

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